1 Oriented surfaces

A surface is called oriented if a side has been chosen. Given an oriented surface $\Sigma \subset \mathbb{R}^3$, it has a "normal" vector field $N : \Sigma \to \mathbb{R}^3$ such that for each point $p \in \Sigma$:

- $N(p)$ is perpendicular to Σ at p
- $N(p)$ points in the preferred direction
- $|N(p)| = 1$

The vector field N is also called an orientation, or the Gauss map of the surface.

Given an oriented surface Σ, we say a parametrization $\varphi : U \to \Sigma$ is compatible with the orientation if at each point, the normal vector

$$\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}(u, v)$$

points in the same direction as $N(\varphi(u, v))$. In that case,

$$N(\varphi(u, v)) = \frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}(u, v)}{|\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}(u, v)|}.$$

Not all surfaces are orientable. For example, a Möbius band has only one side, so it is impossible to "pick one side".

We say a surface is closed if it is compact and does not have boundary. For example, a sphere, an ellipsoid, a torus, and the surface of a polyhedron are closed surfaces. Note that a closed orientable surface in $\mathbb{R}^3$ has an interior and an exterior. We say that it is positively oriented if the unit normal points "outwards" and negatively oriented if the unit normal points "inwards".

2 Surface integrals of vector fields

Definition 1. Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface with a compatible parametrization $\varphi : U \to \Sigma$, and

$$F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

a vector field. Then integral of F over Σ is given by

$$\iint_\Sigma F \cdot dS := \iint_U F(\varphi(u, v)) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right) \, dudv$$

Note that in that case:

$$\begin{aligned}
\iint_{\Sigma} F \cdot dS &= \iint_{U} F(\varphi(u,v)) \cdot \left(\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}\right) \, dudv \\
&= \iint_{U} F(\varphi(u,v)) \cdot \left(\frac{\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}}{|\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}|}\right) |\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}| \, dudv \\
&= \iint_{U} F(\varphi(u,v)) \cdot N(\varphi(u,v)) \, |\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}| \, dudv \\
&= \iint_{\Sigma} (F \cdot N) \, dS,
\end{aligned}$$

Therefore,

$$\iint_{\Sigma} F \cdot dS = \iint_{\Sigma} (F \cdot N) \, dS$$

meaning that the integral of the vector field F is the same as the integral of the scalar function $F \cdot N$.

Note: if you use a parametrization not compatible with the orientation, the integral changes sign.

These are great to model:

- Flow of a fluid across an imaginary boundary.
- Flow of sodium/polasium/glucose/oxygen through the membrane of a cell.
- Electric flux in electromagnetism.
- Integral of $\mathrm{Curl}(F)$ measures amount of rotation of objects over Σ.

Exercise 1 *Let Σ_1 be the square with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$, Σ_2 the square with vertices $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(0,0,1)$, and Σ_3 the square with vertices $(0,0,0)$, $(0,1,0)$, $(0,1,1)$, $(0,0,1)$, all oriented towards the first octant. In other words,*

- *Σ_1 is the unit square in the xy-plane.*
- *Σ_2 is the unit square in the xz-plane.*
- *Σ_3 is the unit square in the yz-plane.*

Let $F = \langle 1,0,0 \rangle$. Then

$$\iint_{\Sigma_1} F \cdot dS = \iint_{\Sigma_2} F \cdot dS = 0$$
$$\iint_{\Sigma_3} F \cdot dS = 1$$

This is because:

- *The unit normal of* Σ_1 *is* $\langle 0,0,1\rangle$.
- *The unit normal of* Σ_2 *is* $\langle 0,1,0\rangle$.
- *The unit normal of* Σ_3 *is* $\langle 1,0,0\rangle$.

When we take the dot product with F*, only the thid one doesn't cancel. In that case, we get*

$$\begin{aligned}\iint_{\Sigma_3} F\cdot dS &= \iint_{\Sigma_3} (F\cdot N)\,dS \\ &= \iint_{\Sigma_3} 1\,dS \\ &= Area(\Sigma_3) \\ &= 1\end{aligned}$$

Exercise 2 *Let* Σ *be the portion of the plane* $x-y+z=4$ *inside the cylinder* $x^2+y^2=9$*, oriented upwards. Compute*

$$\iint_\Sigma \langle 3+z, 2x^2, xy-1\rangle\cdot dS$$

We use the parametrization $\varphi : D\to\Sigma$, where $D\subset\mathbb{R}^2$ is the interior of the circle $x^2+y^2=9$ and φ is given by

$$\varphi(u,v) = (u,v,4-u+v)$$

Then we compute

$$\begin{aligned}\frac{\partial\varphi}{\partial u} &= \langle 1,0,-1\rangle \\ \frac{\partial\varphi}{\partial v} &= \langle 0,1,1\rangle \\ \frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v} &= \langle 1,-1,1\rangle\end{aligned}$$

The cross product points "up" because the third component is positive, so the parametrization is compatible with the orientation. Setting $F(x,y,z) := \langle 3+z, 2x^2, xy-1\rangle$, we get

$$\begin{aligned}F(\varphi(u,v))\cdot\frac{\partial\varphi}{\partial u}\times\frac{\partial\varphi}{\partial v} &= \langle 3+(4-u+v), 2u^2, uv-1\rangle\cdot\langle 1,-1,1\rangle \\ &= 6-u+v-2u^2+uv\end{aligned}$$

Then we compute, using polar coordinates,

$$\begin{aligned}
\iint_\Sigma F \cdot dS &= \iint_D [6 - u + v - 2u^2 + uv]\, dudv \\
&= \int_0^3 \int_0^{2\pi} r[6 - r\cos\theta + r\sin\theta - 2r^2\cos^2\theta + r^2\sin\theta\cos\theta]\, d\theta dr \\
&= \int_0^3 [12\pi r - 2\pi r^3]\, dr \\
&= [6 \cdot 3^2 - \frac{1}{2} \cdot 3^4]\pi \\
&= \frac{27}{2}\pi
\end{aligned}$$

Exercise 3 *Let* Σ *be the triangle with vertices* $(3,0,0)$, $(0,6,0)$, $(0,0,5)$, *oriented downwards. Compute*

$$\iint_\Sigma \langle 3z, y, 2\rangle \cdot dS$$

Note that Σ is part of the plane $10x+5y+6z = 30$. We use the parametrization $\varphi : U \to \Sigma$ with $U \subset \mathbb{R}^2$ the triangle with vertices $(0,0)$, $(3,0)$, $(0,6)$, and

$$\varphi(u,v) := \left(u, v, 5 - \frac{5}{3}x - \frac{5}{6}y\right).$$

Using this change of variables,

$$\langle 3z, y, 2\rangle = \langle 15 - 5u - \frac{5}{2}v, v, 2\rangle$$

Also we compute the Jacobian

$$\begin{aligned}
\frac{\partial\varphi}{\partial u} &= \langle 1, 0, -\frac{5}{3}\rangle \\
\frac{\partial\varphi}{\partial v} &= \langle 0, 1, -\frac{5}{6}\rangle \\
\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v} &= \langle \frac{5}{3}, \frac{5}{6}, 1\rangle
\end{aligned}$$

Since $\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}$ points upwards, it is NOT compatible with the orientation, so we put a minus sign! Then

$$\begin{aligned}
\iint_\Sigma \langle 3z, y, 2\rangle \cdot dS &= -\iint_D \langle 15 - 5u - \frac{5}{2}v, v, 2\rangle \cdot \langle \frac{5}{3}, \frac{5}{6}, 1\rangle \, dvdu \\
&= -\int_0^3 \int_0^{6-2u} [25 - \frac{25}{3}u - \frac{25}{6}v + \frac{5}{6}v + 2] \, dudv \\
&= -\int_0^3 [27(6-2u) - \frac{25}{3}u(6-2u) - \frac{5}{3}(6-2u)^2] \, du \\
&= -\int_0^3 [162 - 54u - 50u + \frac{50}{3}u^2 - 60 + 40u - \frac{20}{3}u^2] \, du \\
&= -\int_0^3 [102 - 64u + 10u^2] \, du \\
&= -[306 - 32 \cdot 3^2 + 10 \cdot 3^2] \\
&= -108
\end{aligned}$$