1 Divergence Theorem

Exercise 1 *Let $\Sigma \subset \mathbb{R}^3$ be the sphere $x^2 + y^2 + z^2 = 9$, negatively oriented. Compute*

$$\iint_\Sigma \langle x^2 + 3z, 2y + xyz, 5z - x \rangle \cdot dS$$

Exercise 2 *Let $\Sigma \subset \mathbb{R}^3$ be the boundary of the box $[-1, 2] \times [0, 3] \times [-2, 1]$, positively oriented. Compute*

$$\iint_\Sigma \langle 2x - yz, 3y + xz, 2yz + e^x \rangle \cdot dS$$

Exercise 3 *Let $\Sigma \subset \mathbb{R}^3$ be the boundary of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$, oriented outward. Compute*

$$\iint_\Sigma \langle xz, yz, x^2 + y^2 \rangle \cdot dS$$

Exercise 4 *Let $\Sigma \subset \mathbb{R}^3$ be the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$, oriented outward. Compute*

$$\iint_\Sigma \langle 2x + yz, 3y + x\cos z, e^x + y^2 \rangle \cdot dS$$

Exercise 5 *Let $\Sigma \subset \mathbb{R}^3$ be the closed surface consisting of the portion of the paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$, together with the disk $x^2 + y^2 \leq 9$ in the xy-plane, oriented outward. Compute*

$$\iint_\Sigma \langle xy + 1, yz - 3, zx + 5y \rangle \cdot dS$$

Exercise 6 *Let $\Sigma \subset \mathbb{R}^3$ be the closed surface consisting of the portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 4$ together with the disk $x^2 + y^2 \leq 16$ in the plane $z = 4$, oriented outward. Compute*

$$\iint_\Sigma \langle xy + 3y, 4y + z, 5z + x \rangle \cdot dS$$

Exercise 7 *Let $\Sigma \subset \mathbb{R}^3$ be the boundary of the tetrahedron with vertices $(3, 0, 0)$, $(0, -3, 0)$, $(0, 0, 2)$, $(0, 0, 0)$, oriented outward. Compute*

$$\iint_\Sigma \langle 3x - 2y + e^z, 5x^2 - y, z - \cos y \rangle \cdot dS$$

Exercise 8 *Let $\Sigma_1 \subset \mathbb{R}^3$ be the disk $x^2 + y^2 \leq 1$ in the xy-plane, oriented upward, and $\Sigma_2 \subset \mathbb{R}^3$ the portion of the sphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$, oriented upward. Consider a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ with positive divergence. Which one is larger?*

$$\iint_{\Sigma_1} F \cdot dS \qquad \text{or} \qquad \iint_{\Sigma_2} F \cdot dS$$

Exercise 9 *Let $\Sigma_1 \subset \mathbb{R}^3$ be the disk $y^2 + z^2 \leq 1$ in the yz-plane, oriented towards the x-axis, and $\Sigma_2 \subset \mathbb{R}^3$ the portion of the sphere $x^2 + y^2 + z^2 = 1$ with $x \leq 0$, oriented towards the x-axis. Consider a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ with negative divergence. Which one is larger?*

$$\iint_{\Sigma_1} F \cdot dS \qquad \text{or} \qquad \iint_{\Sigma_2} F \cdot dS$$

Exercise 10 *Give a proof of the Divergence Theorem in the case where the region is rectangular. That is, let $E = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$, ∂E its boundary oriented positively, and $F(x, y, z)$ a vector field whose domain contains E. Then*

$$\iint_{\partial E} F \cdot dS = \iiint_E \operatorname{div}(F)\, dV$$