1 Green's Theorem

Exercise 1 *Let $C \subset \mathbb{R}^2$ be the curve that travels along straight lines from $(0,0)$ to $(4,0)$, then from $(4,0)$ to $(0,3)$, and then from $(0,3)$ back to $(0,0)$. Compute*

$$\int_C \langle 3x - y^2 - 1, x + 2y\rangle \cdot d\gamma$$

Hint: *The curve is closed and travels counterclockwise, so the integral above equals the integral of $1 + 2y$ (the curl of $\langle 3x - y^2 - 1, x + 2y\rangle$) over the region surrounded by C.*

Exercise 2 *Let $C \subset \mathbb{R}^2$ be the curve that travels along straight lines from $(-1,2)$ to $(1,2)$, then from $(1,2)$ to $(1,3)$, then from $(1,3)$ to $(-1,3)$, and then from $(-1,3)$ back to $(-1,2)$. Compute*

$$\int_C \langle x^2y + 3x - 5, 2xy + e^y + 3\rangle \cdot d\gamma$$

Hint: *The curve is closed and travels counterclockwise, so the integral above equals the integral of $2y - x^2$ (the curl of $\langle x^2y + 3x - 5, 2xy + e^y + 3\rangle$) over the region surrounded by C.*

Exercise 3 *Let $C \subset \mathbb{R}^2$ be the circle $x^2 + y^2 = 9$ oriented counterclockwise. Compute*

$$\int_C \langle xy + 5y^3 + 2, x - ye^y\rangle \cdot d\gamma$$

Exercise 4 *Let $C \subset \mathbb{R}^2$ be the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ oriented counterclockwise. Compute*

$$\int_C \langle -3y,\ 2x + y\rangle \cdot d\gamma$$

Exercise 5 *Let $C \subset \mathbb{R}^2$ be the curve that travels along straight lines from $(-2,4)$ to $(-2,0)$, then from $(-2,0)$ to $(2,0)$, then from $(2,0)$ to $(2,4)$, and then back to $(-2,4)$ along the parabola $y = x^2$. Compute*

$$\int_C \langle 3y + x^2 - 2, 2x + y + 7\rangle \cdot d\gamma$$

Hint: *The curve is closed and travels counterclockwise, so the integral above equals the integral of $2 - 3 = -1$ (the curl of $\langle 3y + x^2 - 2, 2x + y + 7\rangle$) over the region surrounded by C.*

Exercise 6 *Let $C \subset \mathbb{R}^2$ be the curve that travels along straight lines from $(0,0)$ to $(3,1)$, then from $(3,1)$ to $(4,4)$, then from $(4,4)$ to $(1,3)$, and then from $(1,3)$ back to $(0,0)$. Compute*

$$\int_C \langle y - 3x + 2, x^2 - 3y + 5 \rangle \cdot d\gamma$$

Exercise 7 *Let $C_1 \subset \mathbb{R}^2$ be the curve that travels along straight lines from $(0,0)$ to $(3,0)$, then from $(3,0)$ to $(4,4)$. Let $C_2 \subset \mathbb{R}^2$ be the curve that travels along straight lines from $(0,0)$ to $(0,5)$, then from $(0,5)$ to $(4,4)$. Let $C_3 \subset \mathbb{R}^2$ be the straight line from $(0,0)$ to $(4,4)$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field with positive curl. Among the integrals*

$$\int_{C_1} F \cdot d\gamma_1, \qquad \int_{C_2} F \cdot d\gamma_2, \qquad \int_{C_3} F \cdot d\gamma_3,$$

which one is the largest and which one the smallest?

Hint: *The curve C_4 that travels along C_1 and then along C_3 but in the reverse direction, is closed and travels counterclockwise. Therefore*

$$\int_{C_1} F \cdot d_{\gamma_1} - \int_{C_3} F \cdot d_{\gamma_3} = \iint_D curl(F)\, dA$$

where D is the region surrounded by C_4.