Examples

Inverse images of regular values, products, covering spaces, orthogonal group

Example 1. $\mathbb{R}^n$ is a smooth manifold in a canonical way.

Proof $\mathbb{R}^n$ is a metric space, hence Hausdorff. The set of balls with rational radius and center with rational coordinates forms a countable sub-basis of the topology of $\mathbb{R}^n$, so it is second countable.

The identity map $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is a chart. Since the domain is everything, and any chart is compatible with itself, the set containing this chart is a smooth atlas. More specifically, $\mathcal{A} := \{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\}$ is a smooth atlas of the topological manifold $\mathbb{R}^n$.

Therefore, there is a unique smooth structure containing this chart. Note that for each open set $U \subset \mathbb{R}^n$, the inclusion $U \hookrightarrow \mathbb{R}^n$ belongs to this smooth structure. ■

Example 2 (Open sets)**.** Let M be an n-dimensional smooth manifold and $U \subset M$ an open set. Then U is an n-dimensional smooth manifold.

Proof sketch Let $\mathcal{S} = \{(U_i, \varphi_i)\}_{i\in I}$ be the smooth structure of M. Then $\mathcal{S}_U := \{(U \cap U_i, \varphi_i|_{U\cap U_i})\}_{i\in I}$ is a smooth atlas on U (it is actually a smooth structure). ■

Example 3 (Locally a graph)**.** Let $M \subset \mathbb{R}^{n+m}$ be a set that is locally the graph of a smooth function. That is, for each $p \in M$, there is a linear isomorphism $L : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and a smooth function $f : U \to \mathbb{R}^n$ such that $L(p) \in U \times V$ and

$$L(M) \cap (U \times V) = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in U\}.$$

Then M is a smooth manifold.

Proof Let $L : \mathbb{R}$, U, V, and f as above. Define

$$\varphi : M \cap L^{-1}(U \times V) \to U$$

as

$$\varphi := \pi \circ L$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the projection onto the first n coordinates. Notice that

$$\begin{aligned} \varphi(L^{-1}(x, f(x))) &= \pi \circ L \circ L^{-1}(x, f(x)) \\ &= \pi(x, f(x)) \\ &= x. \end{aligned}$$

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Hence the inverse $\varphi^{-1} : U \to M \cap L^{-1}(U \times V)$ is the continuous function

$$\varphi^{-1}(x) = L^{-1}(x, f(x)),$$

showing that φ is a chart. We claim that the set of all charts obtained this way form an atlas. For that purpose, take a linear isomorphism $\hat{L} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, open sets $\hat{U} \subset \mathbb{R}^n$, $\hat{V} \subset \mathbb{R}^m$, and a smooth function $\hat{f} : \hat{U} \to \mathbb{R}^n$ such that

$$\hat{L}(M) \cap (\hat{U} \times \hat{V}) = \{(x, \hat{f}(x)) \in \mathbb{R}^n \times \mathbb{R}^m \,|\, x \in \hat{U}\}.$$

Construct with them the chart

$$\psi : M \times \hat{L}^{-1}(\hat{U} \times \hat{V}) \to \hat{U}$$

given by

$$\psi := \pi \circ \hat{L}.$$

The change of coordinates is then given by

$$\psi \circ \varphi^{-1}(x) = (\pi \circ \hat{L})\,(L^{-1}(x, f(x))),$$

which is the composition of the smooth functions

$$x \mapsto (x, f(x)) \mapsto L^{-1}(x, f(x)) \mapsto \hat{L}(L^{-1}(x, f(x))) \mapsto \pi(\hat{L}(L^{-1}(x, f(x)))).$$

Therefore the charts φ and ψ are compatible, proving our claim. ■

Example 4 (Level sets)**.** Let $\Omega \subset \mathbb{R}^{n+m}$ be an open set and $F \in C^\infty(\Omega; \mathbb{R}^m)$. If for all $p \in F^{-1}(0)$, the differential $d_pF : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is surjective, then the level set $F^{-1}(0) \subset \mathbb{R}^{n+m}$ is a smooth manifold.

Proof By the Implicit Function Theorem, this is covered by the above example ■

Example 5 (Products)**.** Let M and N be smooth manifolds of dimension m and n, respectively. Then $M \times N$ is a smooth manifold of dimension $m + n$ in a canonical way.

Proof For each chart (U, φ) of M and each chart (V, ψ) of N, we consider the map

$$\varphi \otimes \psi : U \times V \to \varphi(U) \times \psi(V) \subset \mathbb{R}^{n+m}$$

given by

$$(\varphi \otimes \psi)(x, y) := (\varphi(x), \psi(y)).$$

Since products of homeomorphisms are homeomorphisms, $\varphi \otimes \psi$ is a chart.

Now consider (U', φ') another chart of M and (V', ψ') another chart of N. With those, we build the chart

$$\varphi' \otimes \psi' : U' \times V' \to \varphi'(U') \times \psi'(V').$$

Notice that

$$(U \times V) \cap (U' \times V') = (U \cap U') \times (V \times V').$$

Then the change of coordinates between $\varphi \otimes \psi$ and $\varphi' \otimes \psi'$ is the function

$$(\varphi' \otimes \psi') \circ (\varphi \otimes \psi)^{-1} : (\varphi \otimes \psi)((U \cap U') \times (V \cap V')) \to (\varphi' \otimes \psi')((U \cap U') \times (V \cap V'))$$

given by

$$\begin{aligned}(\varphi' \otimes \psi') \circ (\varphi \otimes \psi)^{-1}(x, y) &= (\varphi' \otimes \psi')(\varphi^{-1}(x), \psi^{-1} y) \\ &= (\varphi' \circ \varphi^{-1}(x), \psi' \circ \psi^{-1}(y)).\end{aligned}$$

Since the maps $\varphi' \circ \varphi^{-1}$ and $\psi' \circ \psi^{-1}$ are smooth, the change of coordinates is smooth. ■

Example 6 (Covering spaces)**.** Let M be a smooth manifold and $\pi : \tilde{M} \to M$ a covering map with $\tilde{M}$ second countable. Show that $\tilde{M}$ admits a smooth structure for which π is smooth.

Proof Homework. ■

Example 7 (Special linear group)**.** Let $M_n(\mathbb{R})$ denote the space of $n \times n$ real matrices and identify it with $\mathbb{R}^{n^2}$. Then

$$SL(n; \mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$$

is a smooth manifold of dimension $n^2 - 1$.

Proof Homework. ■

Example 8 (Orthogonal group)**.** Given $A \in M_n(\mathbb{R})$, we denote by A^T its transpose. Then

$$O(n) := \{A \in M_n(\mathbb{R}) \mid AA^T = \mathrm{Id}\}$$

is a smooth manifold of dimension $n(n-1)/2$.

Proof Homework. ■