
Smooth manifolds

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Contents

I	Smooth manifolds	4
	Calculus review	5
	Smooth manifolds	8
	Smooth functions	10
	Examples	11
II	Vector fields	14
	Tangent vectors	15
	Tangent bundle	16
	Differentials of smooth maps	17
	Vector fields	18
	Flows	19
III	Vector bundles	20
	Vector bundles and trivializations	21
	Vector bundle constructions	22
	Sections	23
	Tensors	24
	Serre–Swan Theorem	25
	Examples	26
IV	Differential forms	27
	Differential forms	28
	Wedge products	29
	Exterior derivatives	30
	Oriented manifolds	31
	Integrals	32
	Manifolds with boundary	33

Stokes Theorem	34
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V Homework	35
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HW 1	36
HW 2	37
HW 3	38
HW 4	39
HW 5	40

Part I

Smooth manifolds

Calculus review

Partial derivatives, C^k functions, chain rule, inverse function theorem, implicit function theorem.

We begin by reviewing some calculus concepts. A reference for this material is Hubbard, J. H., & Hubbard, B. B. (2015). Vector calculus, linear algebra, and differential forms: a unified approach (pp. 818-pages). Matrix Editions.

Definition 1 (Partial derivatives). Let $U \subset \mathbb{R}^n$ be an open set, $f : U \rightarrow \mathbb{R}$ a function, $p \in U$, and $i \in \{1, \dots, n\}$. The i -th partial derivative of f at p is defined as

$$\partial_i f(p) := \lim_{h \rightarrow 0} \frac{f(p + he_i) - f(p)}{h},$$

whenever the limit exists, where $e_i = (0, \dots, 1, \dots, 0)$ denotes the i -th canonical vector. If this limit exists for all $p \in U$, it defines a function

$$\partial_i f : U \rightarrow \mathbb{R}.$$

Definition 2 (C^k functions). Let $U \subset \mathbb{R}^n$ be an open set and $f : U \rightarrow \mathbb{R}$ a function. We say f is C^0 if it is continuous. Inductively, we say it is C^{k+1} if for each $i \in \{1, \dots, n\}$, the function $\partial_i f : U \rightarrow \mathbb{R}$ exists and is C^k . We say that f is C^∞ if it is C^k for all $k \in \mathbb{N}$. C^∞ functions are called *smooth*.

Notation 1. Let $U \subset \mathbb{R}^n$ be an open set. We denote by $C^k(U)$ the set of C^k functions $f : U \rightarrow \mathbb{R}$. Same with k replaced by ∞ .

Proposition 1 (Classical derivative rules). Let $U \subset \mathbb{R}^n$ open and $f, g \in C^\infty(U)$. Then

$$\begin{aligned} \partial_i(f + g) &= \partial_i f + \partial_i g, \\ \partial_i(fg) &= (\partial_i f)g + f(\partial_i g). \end{aligned}$$

If in addition, $g(x) \neq 0$ for all $x \in U$, then

$$\partial_i \left(\frac{f}{g} \right) = \frac{g\partial_i f - f\partial_i g}{g^2}.$$

Notation 2. For $m \in \mathbb{N}$, we denote by $C^k(U; \mathbb{R}^m)$ the set of functions $f : U \rightarrow \mathbb{R}^m$, where $f = (f_1, \dots, f_m)$ and $f_j \in C^k(U)$ for each $j \in \{1, \dots, m\}$. We also denote

$$\partial_i f = (\partial_i f_1, \dots, \partial_i f_m).$$

Definition 3 (Differential of a function). Let $U \subset \mathbb{R}^n$ be an open set, $f : U \rightarrow \mathbb{R}^m$ a function, and $p \in U$. We say a linear function $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *derivative* or *differential* of f at p if

$$\lim_{h \rightarrow 0} \frac{f(p+h) - f(p) - L(h)}{|h|} = 0.$$

In such a case, we say that f is *differentiable at p* .

Theorem 1 (Smooth implies differentiable). Let $U \subset \mathbb{R}^n$ be an open set. If $f \in C^\infty(U; \mathbb{R}^m)$, then f is differentiable at p for all $p \in U$. Moreover, the differential of f at p is unique and given by

$$d_p f := \begin{pmatrix} \partial_1 f_1(p) & \cdots & \partial_n f_1(p) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(p) & \cdots & \partial_n f_m(p) \end{pmatrix}.$$

Notation 3. Throughout this course, the word “differentiable” will mean “smooth” as in Definition 2, ignoring the concept of “differentiable” from Definition 3. This abuse of notation is common in the literature of smooth manifolds.

Theorem 2 (Chain rule). Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, $f \in C^\infty(U; \mathbb{R}^m)$, $h \in C^\infty(V; \mathbb{R}^\ell)$, and $f(U) \subset V$. Then $h \circ f \in C^\infty(U; \mathbb{R}^\ell)$, and for each $p \in U$, the differential is given by

$$d_p(h \circ f) = (d_{f(p)}h) \circ (d_p f).$$

Definition 4 (Diffeomorphism). Let $U, V \subset \mathbb{R}^n$ be open sets. A bijective smooth function $f : U \rightarrow V$ is called a *diffeomorphism* if the inverse function $f^{-1} : V \rightarrow U$ is smooth.

Theorem 3 (Inverse function theorem). Let $U \subset \mathbb{R}^n$ be open, $f \in C^\infty(U; \mathbb{R}^n)$, and $p \in U$. Assume the linear function $d_p f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible. Then there are open sets $U_0 \subset \mathbb{R}^n$ and $V_0 \subset \mathbb{R}^n$ such that $p \in U_0$, $f(p) \in V_0$, and $f : U_0 \rightarrow V_0$ is a diffeomorphism.

Theorem 4 (Implicit function theorem). Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ be open and $F \in C^\infty(\Omega; \mathbb{R}^m)$. Assume the matrix

$$\begin{pmatrix} \partial_{n+1} F_1 & \cdots & \partial_{n+m} F_1 \\ \vdots & \ddots & \vdots \\ \partial_{n+1} F_m & \cdots & \partial_{n+m} F_m \end{pmatrix}$$

is invertible at a point $(p, q) \in F^{-1}(0) \subset \Omega \subset \mathbb{R}^n \times \mathbb{R}^m$. Then there are open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and $f \in C^\infty(U; \mathbb{R}^m)$ such that $p \in U$, $q \in V$, and

$$F^{-1}(0) \cap (U \times V) = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in U\}.$$

Corollary 1 (Implicit function theorem again). Let $\Omega \subset \mathbb{R}^{n+m}$ be open and $F \in C^\infty(\Omega; \mathbb{R}^m)$. Assume the linear map $d_p F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$ is surjective for some $p \in F^{-1}(0) \subset \Omega$. Then there are open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and $f \in C^\infty(U; \mathbb{R}^m)$ such that after reordering the coordinates of \mathbb{R}^{n+m} , one has $p \in U \times V$ and

$$F^{-1}(0) \cap (U \times V) = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in U\}.$$

Remark 1. For the purposes of this class, the above results were stated for smooth functions, but they have analogues for functions of less regularity:

- Proposition 1 only requires the partial derivatives of f and g to exist.
- Theorem 1 only requires that f is C^1 .
- In Theorem 2, if f and g are C^k , then $g \circ f$ is C^k .
- In Theorem 3, f only needs to be C^k with $k \geq 1$. In that case, f^{-1} is also C^k .
- In Theorem 4, F only needs to be C^k with $k \geq 1$. In that case, f is also C^k .

Smooth manifolds

Charts, atlases, smooth structures, and smooth manifolds.

A reference for this material is Chapter 1 of John M. Lee. Introduction to smooth manifolds. Second edition. Grad. Texts in Math., 218. Springer, New York, 2013. xvi+708pp. ISBN: 978-1-4419-9981-8.

Definition 5 (Topological manifold). Let $n \in \mathbb{N}$. An n -dimensional topological manifold is a second-countable Hausdorff topological space M such that each point $p \in M$ has an open neighborhood homeomorphic to an open set in \mathbb{R}^n .

Definition 6 (Charts and parametrizations). Let M be an n -dimensional topological manifold, $U \subset M$ an open set, and $\varphi : U \rightarrow V$ a homeomorphism with $V \subset \mathbb{R}^n$ open. The pair (U, φ) is called a *chart* and the pair (V, φ^{-1}) is called a *parametrization*. By an abuse of notation, we often call φ a chart.

Definition 7 (Compatible charts). Let M be a topological manifold. We say two charts (U, φ) , (V, ψ) are *compatible* if the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$$

is a diffeomorphism.

Proposition 2. Being compatible is an equivalence relation in the set of charts.

Proof Homework. ■

Definition 8 (Smooth atlas). Let M be a topological manifold. A *smooth atlas* is a collection of compatible charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ with

$$M \subset \bigcup_{i \in I} U_i.$$

Definition 9 (Smooth structure). Let M be a topological manifold. A *smooth structure* is a maximal smooth atlas \mathcal{A} . In this definition, maximal means that any smooth atlas containing \mathcal{A} equals \mathcal{A} .

Proposition 3. Any smooth atlas \mathcal{A} is contained in a unique smooth structure \mathcal{S} . Moreover, \mathcal{S} consists precisely of the charts compatible with all charts in \mathcal{A} .

Proof Homework. ■

Definition 10 (Smooth manifold). A smooth manifold is a pair (M, \mathcal{S}) with M a topological manifold and \mathcal{S} a smooth structure on M .

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Definition 11 (Smooth function to \mathbb{R}^m). Let (M, \mathcal{S}) be a smooth manifold and $f : M \rightarrow \mathbb{R}^m$ a function. We say f is *smooth* if for any chart (U, φ) in \mathcal{S} , the composition

$$f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^m$$

is smooth. We denote by $C^\infty(M)$ the set of smooth functions $f : M \rightarrow \mathbb{R}$.

Proposition 4 (Smoothness is local). Let (M, \mathcal{S}) be a smooth manifold, $\mathcal{A} \subset \mathcal{S}$ a smooth atlas, and $f : M \rightarrow \mathbb{R}^m$ a function. Assume that for any chart (U, φ) in \mathcal{A} , the composition

$$f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^m$$

is smooth. Then f is smooth.

Proof Exercise. Use that the change of coordinates is smooth the chain rule. ■

Notation 4. Given a smooth manifold (M, \mathcal{S}) , by an abuse of notation, whenever we say “chart” we mean an element of \mathcal{S} .

Notation 5. While a smooth manifold is technically a pair (M, \mathcal{S}) , it is often simply denoted by M .

Smooth functions

Smooth functions, diffeomorphisms, bump functions, and partitions of unity.

A reference for this material is Chapter 2 of John M. Lee. Introduction to smooth manifolds. Second edition. Grad. Texts in Math., 218. Springer, New York, 2013. xvi+708pp. ISBN: 978-1-4419-9981-8.

Definition 12 (Smooth function to a manifold). Let M, N be smooth manifolds and $f : M \rightarrow N$ a continuous function. We say f is *smooth* if for any chart (U, φ) of M and any chart (V, ψ) of N , the composition

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V)) \rightarrow \psi(V)$$

is smooth.

Proposition 5. Let M_1, M_2, M_3 be smooth manifolds and $f : M_1 \rightarrow M_2$, $g : M_2 \rightarrow M_3$ smooth functions. Then $g \circ f : M_1 \rightarrow M_3$ is smooth.

Proof Exercise. Use the definition and the chain rule. ■

Definition 13 (Diffeomorphism). Let M, N be smooth manifolds. A smooth homeomorphism $f : M \rightarrow N$ is called a *diffeomorphism* if its inverse $f^{-1} : N \rightarrow M$ is smooth. If there is a diffeomorphism $f : M \rightarrow N$, we say that M is *diffeomorphic* to N .

Proposition 6. Being diffeomorphic is an equivalence relation.

Proof Exercise. Use Proposition ???. ■

Definition 14 (Support). Let M be a topological space and $f : M \rightarrow \mathbb{R}$ a continuous function. The *support* of f , denoted by $\text{supp}(f)$ is defined as the topological closure of the set

$$\{x \in M \mid f(x) \neq 0\}.$$

Proposition 7 (Bump functions). Let M be a smooth manifold, $U \subset M$ an open set, and $p \in U$. Then there is $f \in C^\infty(M)$ with $\text{supp}(f) \subset U$ and $f(p) \equiv 1$ in a neighborhood of p .

Proof Homework. ■

Corollary 2. Let M be a smooth manifold, $U \subset M$ an open set, $p \in U$, and $f \in C^\infty(U)$. Then there is $g \in C^\infty(M)$ with $g \equiv f$ in a neighborhood of p .

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Definition 15 (Locally finite). Let M be a topological space and $\mathcal{X} = \{X_i\}_{i \in I}$ a collection of subsets of M . We say \mathcal{X} is *locally finite* if each $p \in M$ admits an open neighborhood $U \subset M$ such that $U \cap X_i \neq \emptyset$ only for finitely many i 's.

Theorem 5 (Partitions of unity). Let M be a smooth manifold and $\{U_i\}_{i \in I}$ an open cover of M . Then there is a collection $\{\rho_i\}_{i \in I}$ of functions $\rho_i \in C^\infty(M)$ such that

- For each $i \in I$, one has $\text{supp}(\rho_i) \subset U_i$.
- $\rho_i(M) \subset [0, 1]$
- The collection $\{\text{supp}(\rho_i)\}_{i \in I}$ is locally finite.
- For each $x \in M$, one has

$$\sum_{i \in I} \rho_i(x) = 1.$$

Note that the sum in the last item of Theorem 5 makes sense since only finitely many summands are finite.

Proof Theorem 2.23 from Lee. ■

Examples

Inverse images of regular values, products, covering spaces, orthogonal group

Example 1. \mathbb{R}^n is a smooth manifold in a canonical way.

Proof \mathbb{R}^n is a metric space, hence Hausdorff. The set of balls with rational radius and center with rational coordinates forms a countable sub-basis of the topology of \mathbb{R}^n , so it is second countable.

The identity map $\text{Id}_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a chart. Since the domain is everything, and any chart is compatible with itself, the set containing this chart is a smooth atlas. More specifically, $\mathcal{A} := \{(\mathbb{R}^n, \text{Id}_{\mathbb{R}^n})\}$ is a smooth atlas of the topological manifold \mathbb{R}^n .

Therefore, there is a unique smooth structure containing this chart. Note that for each open set $U \subset \mathbb{R}^n$, the inclusion $U \hookrightarrow \mathbb{R}^n$ belongs to this smooth structure. ■

Example 2 (Open sets). Let M be an n -dimensional smooth manifold and $U \subset M$ an open set. Then U is an n -dimensional smooth manifold.

Proof sketch Let $\mathcal{S} = \{(U_i, \varphi_i)\}_{i \in I}$ be the smooth structure of M . Then $\mathcal{S}_U := \{(U \cap U_i, \varphi_i|_{U \cap U_i})\}_{i \in I}$ is a smooth atlas on U (it is actually a smooth structure). ■

Example 3 (Locally a graph). Let $M \subset \mathbb{R}^{n+m}$ be a set that is locally the graph of a smooth function. That is, for each $p \in M$, there is a linear isomorphism $L : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}$, open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and a smooth function $f : U \rightarrow \mathbb{R}^m$ such that $L(p) \in U \times V$ and

$$L(M) \cap (U \times V) = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in U\}.$$

Then M is a smooth manifold.

Proof Let $L : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}$, $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and f as above. Define

$$\varphi : M \cap L^{-1}(U \times V) \rightarrow U$$

as

$$\varphi := \pi \circ L$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ denotes the projection onto the first n coordinates. Notice that

$$\begin{aligned} \varphi(L^{-1}(x, f(x))) &= \pi \circ L \circ L^{-1}(x, f(x)) \\ &= \pi(x, f(x)) \\ &= x. \end{aligned}$$

Hence the inverse $\varphi^{-1} : U \rightarrow M \cap L^{-1}(U \times V)$ is the continuous function

$$\varphi^{-1}(x) = L^{-1}(x, f(x)),$$

showing that φ is a chart. We claim that the set of all charts obtained this way form an atlas. For that purpose, take a linear isomorphism $\hat{L} : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}$, open sets $\hat{U} \subset \mathbb{R}^n$, $\hat{V} \subset \mathbb{R}^m$, and a smooth function $\hat{f} : \hat{U} \rightarrow \mathbb{R}^m$ such that

$$\hat{L}(M) \cap (\hat{U} \times \hat{V}) = \{(x, \hat{f}(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in \hat{U}\}.$$

Construct with them the chart

$$\psi : M \times \hat{L}^{-1}(\hat{U} \times \hat{V}) \rightarrow \hat{U}$$

given by

$$\psi := \pi \circ \hat{L}.$$

The change of coordinates is then given by

$$\psi \circ \varphi^{-1}(x) = (\pi \circ \hat{L})(L^{-1}(x, f(x))),$$

which is the composition of the smooth functions

$$x \mapsto (x, f(x)) \mapsto L^{-1}(x, f(x)) \mapsto \hat{L}(L^{-1}(x, f(x))) \mapsto \pi(\hat{L}(L^{-1}(x, f(x)))).$$

Therefore the charts φ and ψ are compatible, proving our claim. ■

Example 4 (Level sets). Let $\Omega \subset \mathbb{R}^{n+m}$ be an open set and $F \in C^\infty(\Omega; \mathbb{R}^m)$. If for all $p \in F^{-1}(0)$, the differential $d_p F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$ is surjective, then the level set $F^{-1}(0) \subset \mathbb{R}^{n+m}$ is a smooth manifold.

Proof By the Implicit Function Theorem, this is covered by the above example ■

Example 5 (Products). Let M and N be smooth manifolds of dimension m and n , respectively. Then $M \times N$ is a smooth manifold of dimension $m + n$ in a canonical way.

Proof For each chart (U, φ) of M and each chart (V, ψ) of N , we consider the map

$$\varphi \otimes \psi : U \times V \rightarrow \varphi(U) \times \psi(V) \subset \mathbb{R}^{n+m}$$

given by

$$(\varphi \otimes \psi)(x, y) := (\varphi(x), \psi(y)).$$

Since products of homeomorphisms are homeomorphisms, $\varphi \otimes \psi$ is a chart.

Now consider (U', φ') another chart of M and (V', ψ') another chart of N . With those, we build the chart

$$\varphi' \otimes \psi' : U' \times V' \rightarrow \varphi'(U') \times \psi'(V').$$

Notice that

$$(U \times V) \cap (U' \times V') = (U \cap U') \times (V \cap V').$$

Then the change of coordinates between $\varphi \otimes \psi$ and $\varphi' \otimes \psi'$ is the function

$$(\varphi' \otimes \psi') \circ (\varphi \otimes \psi)^{-1} : (\varphi \otimes \psi)((U \cap U') \times (V \cap V')) \rightarrow (\varphi' \otimes \psi')((U \cap U') \times (V \cap V'))$$

given by

$$\begin{aligned} (\varphi' \otimes \psi') \circ (\varphi \otimes \psi)^{-1}(x, y) &= (\varphi' \otimes \psi')(\varphi^{-1}(x), \psi^{-1}(y)) \\ &= (\varphi' \circ \varphi^{-1}(x), \psi' \circ \psi^{-1}(y)). \end{aligned}$$

Since the maps $\varphi' \circ \varphi^{-1}$ and $\psi' \circ \psi^{-1}$ are smooth, the change of coordinates is smooth. ■

Example 6 (Covering spaces). Let M be a smooth manifold and $\pi : \tilde{M} \rightarrow M$ a covering map with \tilde{M} second countable. Show that \tilde{M} admits a smooth structure for which π is smooth.

Proof Homework. ■

Example 7 (Special linear group). Let $M_n(\mathbb{R})$ denote the space of $n \times n$ real matrices and identify it with \mathbb{R}^{n^2} . Then

$$SL(n; \mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$$

is a smooth manifold of dimension $n^2 - 1$.

Proof Homework. ■

Example 8 (Orthogonal group). Given $A \in M_n(\mathbb{R})$, we denote by A^T its transpose. Then

$$O(n) := \{A \in M_n(\mathbb{R}) \mid AA^T = \text{Id}\}$$

is a smooth manifold of dimension $n(n-1)/2$.

Proof Homework. ■

Part II
Vector fields

Tangent vectors

Different definitions of tangent vectors. A reference for this material is

Tangent bundle

The tangent bundle as a smooth manifold. A reference for this material is

Differentials of smooth maps

A reference for this material is

Vector fields

Basic properties of vector fields, including locality. A reference for this material is

Flows

Flows of vector fields: existence and uniqueness. A reference for this material is

Part III
Vector bundles

Vector bundles and trivializations

A reference for this material is

Vector bundle constructions

A reference for this material is

Sections

A reference for this material is

Tensors

A reference for this material is

Serre–Swan Theorem

A reference for this material is

Examples

Part IV

Differential forms

Differential forms

A reference for this material is

Wedge products

A reference for this material is

Exterior derivatives

A reference for this material is

Oriented manifolds

A reference for this material is

Integrals

A reference for this material is

Manifolds with boundary

A reference for this material is

Stokes Theorem

A reference for this material is

Part V

Homework

HW 1

Due April 10.

Exercise 1 Let M be an n -dimensional topological manifold. Show that being compatible is an equivalence relation in the set of charts of M .

Exercise 2 Let M be an n -dimensional topological manifold and \mathcal{A} a smooth atlas. Show that there is a unique smooth structure \mathcal{S} containing \mathcal{A} . Moreover, show that \mathcal{S} consists precisely of the charts compatible with all charts in \mathcal{A} .

Exercise 3 Let M be a smooth manifold and $\pi : \tilde{M} \rightarrow M$ a covering map with \tilde{M} second countable. Show that \tilde{M} admits a smooth structure for which π is smooth.

Hint: Consider a chart (U, φ) of M with $U \subset M$ an evenly covered neighborhood. For each section $\sigma : U \rightarrow \tilde{M}$, consider the chart $(\sigma(U), \varphi \circ \pi)$. Show that the set of charts of \tilde{M} obtained this way is a smooth atlas. Recall that a section is a continuous map $\sigma : U \rightarrow \tilde{M}$ with $\pi \circ \sigma = \text{Id}_U$.

Exercise 4 Let $M_n(\mathbb{R})$ denote the space of $n \times n$ real matrices and identify it with \mathbb{R}^{n^2} . Show that

$$SL(n; \mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$$

is a smooth manifold of dimension $n^2 - 1$.

Exercise 5 Given $A \in M_n(\mathbb{R})$, we denote by A^T its transpose. Show that

$$O(n) := \{A \in M_n(\mathbb{R}) \mid AA^T = 1\}$$

is a smooth manifold of dimension $n(n-1)/2$.

Hint: Identify $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ with $(\mathbb{R}^n)^n$, the set of n -tuples of elements of \mathbb{R}^n , and $\mathbb{R}^{n(n+1)/2}$ with the set of upper-triangular $n \times n$ matrices. Define

$$F : (\mathbb{R}^n)^n \rightarrow \mathbb{R}^{n(n+1)/2}$$

by

$$F_{ij}(\vec{v}_1, \dots, \vec{v}_n) = \vec{v}_i \cdot \vec{v}_j - \delta_{ij}$$

for $1 \leq i \leq j \leq n$, where δ_{ij} denotes the Kronecker delta. Show that

$$F^{-1}(0) = O(n)$$

and that for all $p \in O(n)$ the differential $d_p F : (\mathbb{R}^n)^n \rightarrow \mathbb{R}^{n(n+1)/2}$ is surjective.

HW 2

Due April 25.

HW 3

Due May 8.

HW 4

Due May 22.

HW 5

Due June 5.