Smooth manifolds

Charts, atlases, smooth structures, and smooth manifolds.

A reference for this material is Chapter 1 of John M. Lee. Introduction to smooth manifolds. Second edition. Grad. Texts in Math., 218. Springer, New York, 2013. xvi+708pp. ISBN: 978-1-4419-9981-8.

Definition 1 (Topological manifold)**.** Let $n \in \mathbb{N}$. An *n-dimensional topological manifold* is a second-countable Hausdorff topological space M such that each point $p \in M$ has an open neighborhood homeomorphic to an open set in $\mathbb{R}^n$.

Definition 2 (Charts and parametrizations)**.** Let M be an n-dimensional topological manifold, $U \subset M$ an open set, and $\varphi : U \to V$ a homeomorphism with $V \subset \mathbb{R}^n$ open. The pair (U, φ) is called a *chart* and the pair (V, φ^{-1}) is called a *parametrization*. By an abuse of notation, we often call φ a chart.

Definition 3 (Compatible charts)**.** Let M be a topological manifold. We say two charts (U, φ), (V, ψ) are *compatible* if the map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a diffeomorphism.

Proposition 1. Being compatible is an equivalence relation in the set of charts.

Proof Homework. ∎

Definition 4 (Smooth atlas)**.** Let M be a topological manifold. A *smooth atlas* is a collection of compatible charts $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ with

$$M \subset \bigcup_{i \in I} U_i.$$

Definition 5 (Smooth structure)**.** Let M be a topological manifold. A *smooth structure* is a maximal smooth atlas $\mathcal{A}$. In this definition, maximal means that any smooth atlas containing $\mathcal{A}$ equals $\mathcal{A}$.

Proposition 2. Any smooth atlas $\mathcal{A}$ is contained in a unique smooth structure $\mathcal{S}$. Moreover, $\mathcal{S}$ consists precisely of the charts compatible with all charts in $\mathcal{A}$.

Proof Homework. ∎

Definition 6 (Smooth manifold)**.** A smooth manifold is a pair $(M, \mathcal{S})$ with M a topological manifold and $\mathcal{S}$ a smooth structure on M.

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Definition 7 (Smooth function to $\mathbb{R}^m$)**.** Let $(M, \mathcal{S})$ be a smooth manifold and $f : M \to \mathbb{R}^m$ a function. We say f is *smooth* if for any chart (U, φ) in $\mathcal{S}$, the composition

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$$

is smooth. We denote by $C^\infty(M)$ the set of smooth functions $f : M \to \mathbb{R}$.

Proposition 3 (Smoothness is local)**.** Let $(M, \mathcal{S})$ be a smooth manifold, $\mathcal{A} \subset \mathcal{S}$ a smooth atlas, and $f : M \to \mathbb{R}^m$ a function. Assume that for any chart (U, φ) in $\mathcal{A}$, the composition

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^m$$

is smooth. Then f is smooth.

Proof Exercise. Use that the change of coordinates is smooth the chain rule. ■

Notation 1. Given a smooth manifold $(M, \mathcal{S})$, by an abuse of noation, whenever we say "chart" we mean an element of $\mathcal{S}$.

Notation 2. While a smooth manifold is technically a pair $(M, \mathcal{S})$, it is often simply denoted by M.