Pullbacks

Pullbacks: interaction with wedge product and exterior derivative.

A reference for this material is Chapters 12 and 14 of John M. Lee. Introduction to smooth manifolds. Second edition. Grad. Texts in Math., 218. Springer, New York, 2013. xvi+708pp. ISBN: 978-1-4419-9981-8.

Definition 1 (Pullbacks)**.** Let M, N be smooth manifolds, $f : M \to N$ a smooth function, and $\omega \in \Omega^k(N)$. The *pullback* of ω via f is the k-form $f^*\omega \in \Omega^k(M)$ given by

$$f^*\omega(X_1, \ldots, X_k) := \omega(f_*X_1, \ldots, f_*X_k).$$

Note that if $g \in \Omega^0(N)$ is just a function, then $f^*g = g \circ f$.

Proposition 1 (We love pullbacks)**.** The pullback behaves well with respect to wedge product and exterior derivative. Namely, if $\omega \in \Omega^k(N)$ and $\eta \in \Omega^\ell(N)$, then

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$$

and

$$df^*\omega = f^*d\omega.$$

Solution: For vector fields $X_1, \ldots, X_{k+\ell} \in \mathfrak{X}(M)$, we compute

$$\begin{aligned} f^*(\omega \wedge \eta)(X_1, \ldots, X_{k+\ell}) &= \omega \wedge \eta(f_*X_1, \ldots, f_*X_{k+\ell}) \\ &= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sign}(\sigma)\omega(f_*X_{\sigma(1)}, \ldots, f_*X_{\sigma(k)})\eta(f_*X_{\sigma(k+1)}, \ldots, f_*X_{\sigma(k+\ell)}) \\ &= \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sign}(\sigma)f^*\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})f^*\eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)}) \\ &= (f^*\omega) \wedge (f^*\eta)(X_1, \ldots, X_{k+\ell}). \end{aligned}$$

To prove the second identity, we first prove it for functions. That is, if $\omega \in \Omega^0(M)$, then

$$\begin{aligned} df^*\omega(X) &= d(\omega \circ f)(X) \\ &= X(\omega \circ f) \\ &= f_*X(\omega) \\ &= d\omega(f_*X) \\ &= f^*d\omega(X) \end{aligned}$$

Author(s): Ximera Development Team

Now we work in local coordinates. Let $(x_1, \ldots, x_m)$ be coordinates on M and $(y_1, \ldots, y_n)$ coordinates on N. Recall that

$$f_* \frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

We now do another elementary case:

$$\omega = dy^a.$$

Then

$$f^*\omega = f^* dy^a = \sum_{i=1}^{m} \frac{\partial y^a}{\partial x^i} dx^i.$$

We compute

$$df^*\omega = \sum_{i,j=1}^{m} \frac{\partial^2 y^a}{\partial x^i \partial x^j} dx^i \wedge dx^j.$$

For each pair $i < j$, we know

$$\frac{\partial^2 y^a}{\partial x^i \partial x^j} dx^i \wedge dx^j + \frac{\partial^2 y^a}{\partial x^j \partial x^i} dx^j \wedge dx^i = 0,$$

so

$$df^* dy^a = 0.$$

Now consider

$$\omega = g\, dy^{i_1} \wedge \cdots \wedge dy^{i_k}.$$

Then, using the first property and the Leibniz rule,

$$\begin{aligned}
df^*\omega &= d[(f^*g)\, f^*dy^{i_1} \wedge \cdots f^*dy^{i_k}] \\
&= [d(f^*g)] \wedge f^*dy^{i_1} \wedge \cdots f^*dy^{i_k} \\
&\quad + \sum_{a=1}^{k} (-1)^{a-1} (f^*g)\, f^*dy^{i_1} \wedge \cdots \wedge (df^*dy^{i_a}) \wedge \cdots \wedge f^*dy^{i_k} \\
&= [f^*dg] \wedge f^*dy^{i_1} \wedge \cdots f^*dy^{i_k} \\
&= f^*[dg \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_k}] \\
&= f^* d\omega.
\end{aligned}$$