

Smooth functions

Smooth functions, diffeomorphisms, bump functions, and partitions of unity.

A reference for this material is Chapter 2 of John M. Lee. Introduction to smooth manifolds. Second edition. Grad. Texts in Math., 218. Springer, New York, 2013. xvi+708pp. ISBN: 978-1-4419-9981-8.

Definition 1 (Smooth function to a manifold). Let M, N be smooth manifolds and $f : M \rightarrow N$ a continuous function. We say f is *smooth* if for any chart (U, φ) of M and any chart (V, ψ) of N , the composition

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V)) \rightarrow \psi(V)$$

is smooth.

Proposition 1. Let M_1, M_2, M_3 be smooth manifolds and $f : M_1 \rightarrow M_2$, $g : M_2 \rightarrow M_3$ smooth functions. Then $g \circ f : M_1 \rightarrow M_3$ is smooth.

Proof Exercise. Use the definition and the chain rule. ■

Definition 2 (Diffeomorphism). Let M, N be smooth manifolds. A smooth homeomorphism $f : M \rightarrow N$ is called a *diffeomorphism* if its inverse $f^{-1} : N \rightarrow M$ is smooth. If there is a diffeomorphism $f : M \rightarrow N$, we say that M is *diffeomorphic* to N .

Proposition 2. Being diffeomorphic is an equivalence relation.

Proof Exercise. Use Proposition ???. ■

Definition 3 (Support). Let M be a topological space and $f : M \rightarrow \mathbb{R}$ a continuous function. The *support* of f , denoted by $\text{supp}(f)$ is defined as the topological closure of the set

$$\{x \in M \mid f(x) \neq 0\}.$$

Proposition 3 (Bump functions). Let M be a smooth manifold, $U \subset M$ an open set, and $p \in U$. Then there is $f \in C^\infty(M)$ with $\text{supp}(f) \subset U$ and $f(p) \equiv 1$ in a neighborhood of p .

Proof Homework. ■

Corollary 1. Let M be a smooth manifold, $U \subset M$ an open set, $p \in U$, and $f \in C^\infty(U)$. Then there is $g \in C^\infty(M)$ with $g \equiv f$ in a neighborhood of p .

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Definition 4 (Locally finite). Let M be a topological space and $\mathcal{X} = \{X_i\}_{i \in I}$ a collection of subsets of M . We say \mathcal{X} is *locally finite* if each $p \in M$ admits an open neighborhood $U \subset M$ such that $U \cap X_i \neq \emptyset$ only for finitely many i 's.

Theorem 1 (Partitions of unity). Let M be a smooth manifold and $\{U_i\}_{i \in I}$ an open cover of M . Then there is a collection $\{\rho_i\}_{i \in I}$ of functions $\rho_i \in C^\infty(M)$ such that

- For each $i \in I$, one has $\text{supp}(\rho_i) \subset U_i$.
- $\rho_i(M) \subset [0, 1]$
- The collection $\{\text{supp}(\rho_i)\}_{i \in I}$ is locally finite.
- For each $x \in M$, one has

$$\sum_{i \in I} \rho_i(x) = 1.$$

Note that the sum in the last item of Theorem 1 makes sense since only finitely many summands are finite.

Proof Theorem 2.23 from Lee. ■